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Some Exact Results in Kinetic Theory

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It is shown that for fluids composed of particles interacting with pairwise-additive, spherically symmetric forces, the exact linearized transport equation admits mass, momentum, and kinetic energy as homogeneous solutions and that the kinetic part of the bulk viscosity is identically zero.

KEY WORDS: Exact results in kinetic theory; bulk viscosity; Chapman–Enskog procedure.

In fluids composed of monatomic particles interacting with pairwise-additive, centrally symmetric forces, the exact transport equation[1] for the singlet distribution function f may be linearized by the Chapman–Enskog[2] scheme to give [1,3]

$$d_t^0 f^{(0)} + \mathscr{B}(f^{(0)}) = \mathscr{C}(\phi) \tag{1}$$

where

$$d_t^0 \equiv \partial_t + \mathbf{v}_1 \cdot \nabla_1 + \mathbf{F}^{(0)}(\mathbf{x}_1, t) \cdot \partial_1 \tag{2}$$

$$\mathbf{F}^{(0)}(\mathbf{x}_1, t) = -\nabla_1 \int V(|\mathbf{x}_1 - \mathbf{x}_2|) f^{(0)}(\mathbf{x}_2, \mathbf{v}_2, t)\, d\mathbf{x}_2\, d\mathbf{v}_2 \tag{3}$$

$$\begin{aligned}\mathscr{B}(f^{(0)}) = \mathscr{L} \sum_{n=2}^{N} i \int d\Gamma^{s-1}\, \delta^{s-1}(\mathbf{x}_1 - \mathbf{x}_i)\, \psi^{(s)}(\{-i\nabla\}; i0 + i\partial_t) \\ \times \prod_{j=1}^{s} f^{(0)}(\mathbf{x}_j, \mathbf{v}_j, t)\end{aligned} \tag{4}$$

$$\mathscr{C}(\phi) = \sum_{s=2}^{N} i \int d\mathbf{v}^{s-1}\, \psi^{(s)}(\{0\}; i0) \prod_{j=1}^{s} f^{(0)}(\mathbf{x}_1, \mathbf{v}_j, t) \sum_{\alpha=1}^{s} \phi(\mathbf{x}_1, \mathbf{v}_\alpha, t) \tag{5}$$

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In these expressions, $f^{(0)}$ is the solution to the zeroth-order Chapman–Enskog equation and is of the form⁽¹⁾

$$f^{(0)}(\mathbf{x}, \mathbf{v}, t) = n(m/2\pi kT)^{3/2} \exp[-(m/2kT)(\mathbf{v} - \mathbf{u})^2] \tag{6}$$

n, T and $\mathbf{u}$ are velocity-independent parameters which depend on $\mathbf{x}$ and t and can be identified respectively with the local hydrodynamic density, temperature, and mass-average velocity at a later stage in the theory. The quantity ϕ, defined as $\phi = (f/f^{(0)}) - 1$, accounts for the deviation of f from its "local equilibrium" value $f^{(0)}$. The abbreviations in Eq. (2) are $\partial_t = \partial/\partial t$, $\nabla_1 = \partial/\partial \mathbf{x}_1$, and $\partial_1 = \partial/\partial \mathbf{p}_1$, where t, $\mathbf{x}_1$, and $\mathbf{p}_1$ are the time and the position and momentum of particle 1, respectively. Finally, the quantity $\psi^{(s)}(\{\mathbf{k}\}; z)$ is the collision operator derived by Severne⁽¹⁾:

$$\psi^{(s)}(\{\mathbf{k}\}\,; z) = c^{-s+1} \sum_{n=2}^{\infty} \langle \mathbf{k}_1 + \mathbf{k}_2 + \cdots \mathbf{k}_s \,|(-\delta L)[(L_0 - z)^{-1} (-\delta L)]^n$$
$$\times \mid \mathbf{k}_1 , \mathbf{k}_2 ,\ldots, \mathbf{k}_s\rangle_s^{ir} \tag{7}$$

The quantity c is the mean density of the system. The angular brackets in Eq. (7) denote the Fourier representation of the enclosed operator, which is composed of the free-particle part L_0 and the interaction part δL of the Liouville operator. The collision operator in $\mathscr{B}(f^{(0)})$ is obtained by explicitly evaluating Eq. (7) as a function of $\mathbf{k}_1 ,\ldots, \mathbf{k}_s$ and z and then setting $\mathbf{k}_i = -i\nabla_i$ and $z = i0 + i\partial_t$. The symblol $\mathscr{L}$ in $\mathscr{B}(f^{(0)})$ means that only terms of first order in time and spatial derivatives of n, $\mathbf{u}$, and T are retained. To obtain the collision operator in $\mathcal{C}(\phi)$, the quantities $\mathbf{k}_i$ and z in Eq. (7) are set equal to 0 and $i0$, respectively. The superscript "ir" on the right-hand angular bracket of Eq. (7) indicates that the matrix elements are of the irreducible type defined by Severne⁽¹⁾ and the subscript s means that the enclosed operator involves a set of exactly s particles. Finally, it should be noted that it has been assumed that a transport limit exists, i.e., that the destruction term (arising from initial conditions) vanishes in the long time limit. For a discussion of this point and of the nature of $\psi^{(s)}$, the reader is referred to the original work of Severne.⁽¹⁾

The main points of this article can be summarized as the following theorems valid for fluids composed of particles interacting with pairwise-additive, centrally symmetric forces:

Theorem 1. $\phi_h = 1$, $\mathbf{v}$, and v^2 are solutions of the homogeneous equation

$$\mathcal{C}(\phi_h) = 0 \tag{8}$$

Theorem 2. The kinetic contribution to the bulk viscosity is identically zero.

Theorems 1 and 2 are well known for gases sufficiently dilute that Boltzmann's equation holds. However, the proof given here is true for a fluid of arbitrary density.

An important consequence of Theorem 1 is that it enables one to identify the constants n, $\mathbf{u}$, and T appearing in $f^{(0)}$ with the local hydrodynamic density, mass-average velocity, and temperature, respectively. Thus, the local hydrodynamic variables are determined completely by $f^{(0)}$, and the linear kinetic constitutive relations (i.e., the kinetic fluxes of energy and momentum linear in gradients of the temperature and the hydrodynamic velocity) are determined uniquely by $f^{(0)}\phi$.

To prove Theorem 1, we shall use the following result:

$$\begin{aligned} A &\equiv \sum_{s=2}^{N} i \int d\Gamma^{s-1}\, \psi^{(s)}(\{0\}\,;\, i0) \prod_{j=1}^{s} f^{(0)}(\mathbf{x}_1\,, \mathbf{v}_i\,, t) \\ &= \int d\mathbf{v}_2\, i\partial_1 \cdot \sum_{\mathbf{l}} \langle 0 \,|\, \mathbf{F}_{12} \,|\, \mathbf{l}, -\mathbf{l}\rangle \sum_{s=2}^{N} iNc^{-s+1} \\ &\quad \times \sum_{n=2}^{\infty} \int d\mathbf{v}^{s-2} \langle \mathbf{l}, -\mathbf{l} \,|\, [(L_0 - i0)^{-1}\, (-\delta L)]^{n-1} \,|\, 0\rangle \prod_{j=1}^{s} f^{(0)}(\mathbf{x}_1\,, \mathbf{v}_j\,, t) = 0 \end{aligned} \tag{9}$$

Equation (9) is, in fact, the zeroth-order Chapman–Enskog equation for $f^{(0)}$. The solution has already been given above by Eq. (6). That Eq. (9) is zero for $f^{(0)}$ given by Eq. (6) follows from the fact[1] that the quantity appearing to the right of the matrix element of the force between particles 1 and 2 (i.e., to the right of $\langle 0 \,|\, \mathbf{F}_{12} \,|\, \mathbf{l}, -\mathbf{l}\rangle$) is equal to the Fourier transform of

$$[g_2^{(0)}(\mathbf{x}_1 - \mathbf{x}_2\,; \mathbf{x}_1\,, t) - 1] f^{(0)}(\mathbf{x}_1\,, \mathbf{v}_1\,, t) f^{(0)}(\mathbf{x}_1\,, \mathbf{v}_2\,, t)$$

where $g_2^{(0)}$ is the equilibrium radial distribution function evaluated at the density $n(\mathbf{x}_1\,, t)$ and temperature $T(\mathbf{x}_1\,, t)$. Thus, the quantity appearing to the right of ∂_1 in Eq. (9) is proportional to the average of $\mathbf{F}_{12}$ over an equilibrium ensemble. Since this force is zero, Eq. (9) is identically zero.

We can now prove Theorem 1 by noting that $\psi^{(s)}$ is independent of $n(\mathbf{x}_1\,, t)$, $\beta(\mathbf{x}_1\,, t) = 1/kT(\mathbf{x}_1\,, t)$, and $\mathbf{u}(\mathbf{x}_1\,, t)$. Therefore, we have the identities

$$\mathcal{A}(\phi_h = 1) = n(\mathbf{x}_1\,, t)\, \partial A/\partial n(\mathbf{x}_1\,, t) \tag{10}$$

$$\mathcal{A}(\phi_h = \mathbf{v} - \mathbf{u}) = [1/m\beta(\mathbf{x}_1\,, t)]\, \partial A/\partial \mathbf{u}(\mathbf{x}_1\,, t) \tag{11}$$

$$\begin{aligned} \mathcal{A}(\phi_h = \tfrac{1}{2}m(\mathbf{v} - \mathbf{u})^2) = {} & -[\partial A/\partial\beta(\mathbf{x}_1\,, t)] + \tfrac{3}{2}[n(\mathbf{x}_1\,, t)/\beta(\mathbf{x}_1\,, t)] \\ & \times [\partial A/\partial n(\mathbf{x}_1\,, t)] \end{aligned} \tag{12}$$

Since A was shown to be identically zero for arbitrary $n(\mathbf{x}_1, t)$, $T(\mathbf{x}_1, t)$, and $\mathbf{u}(\mathbf{x}_1, t)$, the right-hand sides of Eqs. (10)–(12) are zero, thus proving the theorem.

Theorem 2 now follows directly from Theorem 1. Since 1, $\mathbf{v}$, and v^2 are homogeneous solutions to Eq. (1), we are free to require the auxiliary conditions on ϕ:

$$\int d\mathbf{v} f^{(0)} \phi \phi_h = 0 \tag{13}$$

for $\phi_h = 1$, $\mathbf{v} - \mathbf{u}$, and $(\mathbf{v} - \mathbf{u})^2$. These conditions allow, as already mentioned, the identification of the n, $\mathbf{u}$, and T appearing in $f^{(0)}$ with the local density, hydrodynamic velocity, and temperature defined by the moments

$$\int f\alpha \, d\mathbf{v} \tag{14}$$

where $\alpha = 1$, $\mathbf{v}/n$, and $(m/3kn)(\mathbf{v} - \mathbf{u})^2$, respectively.

The kinetic part of the bulk viscosity is proportional to the trace of the quantity

$$\int f^{(0)} \phi m(\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) \, d\mathbf{v} \tag{15}$$

The trace of Eq. (15) is identically zero by the auxiliary condition for $\phi_h = (\mathbf{v} - \mathbf{u})^2$. Thus, Theorem 2 is proved. It is easy to prove Theorem 2 for mixtures.

It has previously been noted by Nicolis and Severne(4) that condition (13) for $\phi_h = (\mathbf{v} - \mathbf{u})^2$ implies Theorem 2. These authors, however, did not prove Theorem 1, which is necessary to establish the legitimacy of condition (13).

Nicolis and Severne(4) have pointed out that the kinetic temperature defined by (14) with condition (13) is not identical to the kinetic temperature that one deduces from the assumptions of autocorrelation function theory. It has been shown,(3) however, that, to second order in the interaction potential (i.e., the weak coupling limit), the hydrodynamic equation generated by Eq. (1) for the quantity T appearing in $f^{(0)}$ is identical to the equation for the thermodynamic temperature obtained by applying the assumption of local equilibrium to the macroscopic hydrodynamic equation for the total energy. On the basis of this result, we feel that the kinetic temperature defined here is the one appropriate to linear hydrodynamics.

In the dilute-gas limit, the operator $\mathcal{C}$ becomes the linearized Boltzmann operator, and the fact that mass, momentum, and kinetic energy are solutions to the homogeneous equation $\mathcal{C}(\phi) = 0$ follows because these quantities are invariants of the binary collisional process defining $\mathcal{C}$. Thus, it is interesting that in the general case considered in Theorem 1, the homogeneous equation still has the same solutions, but not for the same reason, since the general operator describes a complicated many-body collisional process.

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